

# On the spectrum of relativistic Schroedinger equation in finite differences.

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## Abstract

We develop a method for constructing asymptotic solutions of finite - difference equations and implement it to a relativistic Schroedinger equation which describes motion of a selfgravitating spherically symmetric dust shell. Exact mass spectrum of black hole formed due to the collapse of the shell is determined from the analysis of asymptotic solutions of the equation.

# 1 Relativistic Schroedinger equation in finite differences

In this note we develop a method for determining the energy spectrum of relativistic Schroedinger equation in finite differences

$$\Psi(m, \mu, S + i\zeta) + \Psi(m, \mu, S - i\zeta) = (F_{in}F_{out})^{-1/2} \left( F_{in} + F_{out} - M^2/4m^2S \right) \Psi(m, \mu, S) \quad (1)$$

which appear to describe a quantum black hole model with selfgravitating dust shell introduced in [1]. Here parameter  $M$  is the bare mass of the shell,  $m = m_{out}$  is the Schwarchild mass of the space-time outside the collapsing shell,  $\mu = m_{in}/m_{out}$  is the ratio of Schwarchild masses inside and outside the shell,  $S = R^2/4G^2m^2$  ( $G$  is gravitational constant) is the dimensionless area of the shell surface and  $F_{in,out} = 1 - 2Gm_{in,out}/R$  are functions entering the Schwarchild line element.

Parameter  $\zeta = m_{pl}/(2m^2)$  where  $m_{pl} = \sqrt{c\hbar/G}$  is Planck mass ( $\hbar$  is Planck constant and  $c$  is the speed of light). It becomes small in semiclassical ( $\hbar \rightarrow 0$ ) limit. In this limit the finite shift in the argument of the wave function in (1) become small and one could use an approximate expression

$$\Psi(S \pm i\zeta) \approx \Psi(S) \pm i\zeta\Psi'(S) - \frac{\zeta^2}{2}\Psi''(S) \quad (2)$$

Then the equation 1 becomes just a usual Schroedinger of nonrelativistic quantum mechanics:

$$-\zeta^2\Psi'' + \left\{ 2 - (F_{in}F_{out})^{-1/2} \left( F_{in} + F_{out} - M^2/4m^2S \right) \right\} \Psi = 0 \quad (3)$$

This equation was studied in [1, 2] in details. There was an expression for the energy spectrum of bound states found and quasiclassical behaviour of the wave function studied.

In this paper we will analyse the solutions of the exact equation (1).

The shift of the argument in the wave function is along the imaginary line, so the equation is naturally defined over some complex one dimensional manifold. The function

$$(F_{in}F_{out})^{1/2} = \sqrt{\left(1 - \frac{1}{\rho}\right) \left(1 - \frac{\mu}{\rho}\right)} = \frac{\sqrt{(\rho - 1)(\rho - \mu)}}{\rho} \quad (4)$$

( $\rho = \sqrt{S}$ ) which enter the equation (1) is a branching function of its argument. So (1) is defined over the Riemannian surface  $S_F$  for the function (4) which is a two dimensional sphere glued from the two Riemannian spheres  $S_+$  and  $S_-$  along the sides of the cut made along the interval  $(\mu, 1)$  of the real line.

There are two points  $\rho = \infty$  on the Riemannian surface  $S_F$  in  $S_+$  and  $S_-$  components, which are singular points of equation (1). They differ by the different choices of the sign of the function  $(F_{in}F_{out})^{1/2}$ .

The natural requirements on the wave function  $\Psi$  of a bound state is that it must decrease with  $\rho \rightarrow \infty$ . But in our case we have two infinities in  $S_+$  and  $S_-$  components of  $S_F$ . As it was argued in [1, 2, 3] the two possibilities for  $\rho \rightarrow \infty$  have the following physical meaning. If one consider maximally extended spherically symmetric solution of Einstein equations in vacuum – Kruskal manifold [4], one finds that the radial coordinate  $\rho$  take the same values on different sides of Einstein-Rosen bridge (in  $R_+$  and  $R_-$  regions of Kruskal manifold [1]). So  $\rho$  could tend to infinity twofold: either in  $R_+$  region or in  $R_-$  region. This complicated structure of Kruskal manifold is mirrored in complicated structure of configuration space of quantum system (the real section of  $S_F$ ) which describe a selfgravitating dust shell. So the two possibilities for  $\rho \rightarrow \infty$  in  $S_+$  and  $S_-$  components of  $S_F$  corresponds to the possibility for  $\rho \rightarrow \infty$  in  $R_+$  and  $R_-$  regions of Kruskal space-time.

But the requirements for the wave function to vanish at infinities are not all the requirements which a reasonable wave function of a physical state must satisfy. Besides it must be one-valued function.

This requirement is satisfied automatically if we consider Schroedinger equation over a complex plane (or over the real line). But the equation (1) is defined over (the real section of) the Riemannian surface  $S_F$ . The two infinities in  $S_{\pm}$  components are both singular points of equation. So the equation (1) is defined in fact over complex manifold which is two-dimensional sphere minus two points – a cylinder. But a cylinder possess a nontrivial cycle. The monodromy of solution along this nontrivial cycle is not identity in general. Therefore, the requirement for the wave function to be one-valued function on the Riemannian surface  $S_F$  (and on its real section) become a nontrivial one. In what follows we will find the asymptotics of exact solutions of equation (1) at both infinities. The analysis of analytical properties of asymptotics and the requirement for one-valued solutions to decrease at infinities will lead us to defining of the mass spectrum for the equation (1)

## 2 Asymptotic solutions

Let us find the asymptotic solutions of this equation at infinities in  $S \pm$  components of  $S_F$ .

It is convenient to rewrite (1) in matrix form.

If we truncate the Tailor expansion

$$\Psi(S - i\zeta) = \Psi + \sum_{k=1}^{2N} \frac{(-i\zeta)^k}{k!} \Psi^{(k)} \quad (5)$$

on  $2N$ -th item and expand asymptotically

$$\begin{aligned} & \left(1 - \frac{\mu}{\sqrt{S}}\right)^{-1/2} \left(1 - \frac{1}{\sqrt{S}}\right)^{-1/2} \approx \\ & 1 + \frac{1+\mu}{2} \frac{1}{\sqrt{S}} + \frac{3\mu^2 + 2\mu + 3}{8} \frac{1}{S} + \frac{5\mu^3 + 3\mu^2 + 3\mu + 5}{16} \frac{1}{S^{3/2}} \end{aligned} \quad (6)$$

in powers of  $S$ , we could rewrite (1) as

$$\begin{aligned} \Psi^{2N} = & -\frac{(2N)!}{(i\zeta)^{2N}} \sum_{k=1}^{N-1} \frac{(i\zeta)^{2k}}{(2k)!} \Psi^{(2k)} + \\ & \frac{(2N)!}{(i\zeta)^{2N}} \left( -1 \pm 1 \pm \frac{(1-\mu)^2 - M^2/m^2}{8S} \pm \right. \\ & \left. \frac{(1+\mu)(2(1-\mu)^2 - M^2/m^2)}{16S^{3/2}} \right) \Psi \end{aligned} \quad (7)$$

where the upper signs correspond to  $S_+$  and the lower ones to  $S_-$  components. It is convenient to introduce a vector function

$$Y_1 = \Psi; \quad Y_2 = \Psi'; \quad \dots; \quad Y_{2N} = \Psi^{2N-1} \quad (8)$$

Then (7) takes the form

$$Y' = AY \quad (9)$$

where

$$A(S) = A_0 + \frac{1}{\sqrt{S}} A_1 + \frac{1}{S} A_2 + \frac{1}{S^{3/2}} A_3 \quad (10)$$

where  $A_0$  has the following form:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 \\ a_1 & 0 & a_3 & 0 & \dots & a_{2N-1} & 0 \end{pmatrix} \quad (11)$$

with coefficients

$$\begin{aligned} a_1 &= \frac{(2N)!}{(i\zeta)^{2N}}(-1 \pm 1) \\ a_{2k} &= 0 \\ a_{2k+1} &= -\frac{(2N)!}{(2k!)(i\zeta)^{2N-2k}}, \quad k > 0 \end{aligned} \quad (12)$$

Matrices  $A_{1,2,3}$  are

$$A_1 = 0 \quad (13)$$

and

$$A_2 = \alpha \begin{pmatrix} 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}; \quad \alpha = \pm \frac{(2N)!}{(i\zeta)^{2N}} \frac{((1-\mu)^2 - M^2/m^2)}{8} \quad (14)$$

$$A_3 = \beta \begin{pmatrix} 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}; \quad \beta = \pm \frac{(2N)!}{(i\zeta)^{2N}} \frac{(1+\mu)(2(1-\mu)^2 - M^2/m^2)}{16} \quad (15)$$

After the change of variable  $S = \rho^2$  equation (9) takes the form

$$\frac{1}{\rho} Y'_\rho = \tilde{A} Y; \quad \tilde{A}(\rho) = \tilde{A}_0 + \frac{1}{\rho^2} \tilde{A}_2 + \frac{1}{\rho^3} \tilde{A}_3 \quad (16)$$

with matrices

$$\tilde{A}_i = 2A_i \quad (17)$$

If we let  $N \rightarrow \infty$  then matrix equation (9) is equivalent to finite - difference equation (1) in asymptotic regions. There exist well-developed method for finding asymptotic solutions ( $\rho \rightarrow \infty$ ) of matrix equations of (9) type [5].

First we need to make a transformation

$$\begin{aligned}\tilde{\tilde{Y}} &= TY \\ \tilde{\tilde{A}} &= T\tilde{A}T^{-1} \\ \tilde{\tilde{Y}}'_\rho &= \tilde{\tilde{A}}\tilde{\tilde{Y}}\end{aligned}\tag{18}$$

such that matrix  $\tilde{\tilde{A}}_0$  becomes diagonal:

$$\tilde{\tilde{A}}_0 = \text{diag}(\lambda_1, \dots, \lambda_{2N})\tag{19}$$

The eigen values  $\lambda_1, \dots, \lambda_{2N}$  could be found by equating the characteristic polynomial for matrix  $\tilde{\tilde{A}}_0$  to zero:

$$P_{2N} = |\tilde{\tilde{A}}_0 - \lambda I| = 0\tag{20}$$

As a result we get an equation

$$P_{2N} = -\frac{(2N)!}{2(i\zeta)^{-2N}} \left( \sum_{k=0}^N \frac{1}{(2k)!} \left( \frac{i\zeta\lambda}{2} \right)^{2k} \pm 1 \right) = 0\tag{21}$$

In the limit  $N \rightarrow \infty$  this equation is equivalent to the following one

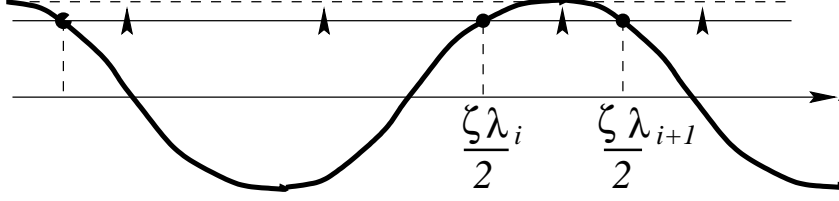
$$\cos\left(\frac{\zeta\lambda}{2}\right) = \pm 1\tag{22}$$

So we see that all the eigen values become double degenerate in the limit  $N \rightarrow \infty$  as it is seen from Fig. 1. This means that the matrix  $\tilde{\tilde{A}}_0$  could not be transformed to diagonal form in this limit, instead it could be transformed to Jordanian form:

$$\begin{aligned}\mathcal{A}_0 &= T\tilde{\tilde{A}}_0T^{-1} = J_1 \oplus J_2 \oplus \dots \oplus J_N; \\ J_i &= \lambda_i I + H; \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\end{aligned}\tag{23}$$

It is convenient to rewrite all the matrices in the form of blocks of dimension  $2 \times 2$ :

$$T = \begin{pmatrix} T_{11} & \dots & T_{1N} \\ \vdots & \dots & \vdots \\ T_{N1} & \dots & T_{NN} \end{pmatrix}; \quad T_{IK} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\tag{24}$$



$$\cos\left(\frac{\zeta \lambda}{2}\right) \rightarrow 1$$

Figure 1: *Double degeneracy of eigen values in  $N \rightarrow \infty$  limit*

and in analogous way

$$\tilde{A}_0 = \begin{pmatrix} A_{11} & \dots & A_{1N} \\ \cdot & \dots & \cdot \\ A_{N1} & \dots & A_{NN} \end{pmatrix}; \quad (25)$$

$$A_{II} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}; \quad A_{I,I+1} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}; \quad I = 1, \dots, N-1$$

$$A_{NK} = \begin{pmatrix} 0 & 0 \\ 2a_{2K-1} & 0 \end{pmatrix}; \quad K = 1, \dots, N-1; \quad A_{NN} = \begin{pmatrix} 0 & 2 \\ 2a_{2N-1} & 0 \end{pmatrix}$$

Then the matrix  $T$  could be found from the equation (23) which lead to the following system of recurrent equations:

$$A_{II}T_{IJ} + A_{I(I+1)}T_{(I+1)J} = \lambda_I T_{IJ} + T_{IJ}H \quad (26)$$

Solving this system of equations gives  $T$  in the following form

$$T_{IJ} = \begin{pmatrix} \left(\frac{\lambda_I}{2}\right)^{2I-2} t_{11}^{(J)} & \left(\frac{\lambda_I}{2}\right)^{2I-2} t_{12}^{(J)} + (I-1) \left(\frac{\lambda_I}{2}\right)^{2I-3} t_{11}^{(J)} \\ \left(\frac{\lambda_I}{2}\right)^{2I-1} t_{11}^{(J)} & \left(\frac{\lambda_I}{2}\right)^{2I-1} t_{12}^{(J)} + \frac{2I-1}{2} \left(\frac{\lambda_I}{2}\right)^{2(I-1)} t_{11}^{(J)} \end{pmatrix} \quad (27)$$

the coefficients  $t_{11}^{(J)}$  and  $t_{12}^{(J)}$  are arbitrary and the form of asymptotic solution of equation (9) do not depend on them. So we will take  $t_{11}^{(J)} = t_{12}^{(J)} = 1$  for simplicity.

Matrices  $\tilde{A}_{2,3}$  transform under (18) as follows:

$$\begin{aligned}\tilde{A}_2 &= \alpha \begin{pmatrix} L_{11} & \dots & L_{1N} \\ \cdot & \dots & \cdot \\ L_{N1} & \dots & L_{NN} \end{pmatrix}; \\ L_{IK} &= \begin{pmatrix} \tilde{t}_{12}^I & \tilde{t}_{12}^I \\ \tilde{t}_{22}^I & \tilde{t}_{22}^I \end{pmatrix}\end{aligned}\quad (28)$$

where  $\tilde{t}_{ij}^I$  are elements of the block  $(T^{-1})_{IN}$  of inverse matrix of transformation  $T^{-1}$ .

The transformed matrix equation (18) could be reduced to a set of  $2 \times 2$  matrix equations with the help of one more transformation

$$\tilde{Y} = P(\rho)Z \quad (29)$$

where

$$P = \begin{pmatrix} 0 & P_{12} & \dots & P_{1N} \\ P_{21} & 0 & \dots & P_{2N} \\ \cdot & \cdot & \dots & \cdot \\ P_{N1} & P_{N2} & \dots & 0 \end{pmatrix} \quad (30)$$

is a matrix of  $2 \times 2$  blocks such that after the transformation (29) the system of equations (9) is rewritten as

$$\frac{1}{\rho}Z' = BZ \quad (31)$$

with matrix

$$B = P^{-1}\tilde{A}P - \frac{1}{\rho}P^{-1}P' \quad (32)$$

of block-diagonal form:

$$B(\rho) = \begin{pmatrix} B_{11} & 0 & \dots & 0 \\ 0 & B_{22} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & B_{NN} \end{pmatrix} \quad (33)$$

Expanding

$$\begin{aligned}P &= P_0 + \frac{1}{\rho}P_1 + \frac{1}{\rho^2}P_2 + \dots; \\ B &= B_0 + \frac{1}{\rho}B_1 + \frac{1}{\rho^2}B_2 + \dots;\end{aligned}\quad (34)$$



and solving equation (32) in each power of  $1/\rho$  one obtains for  $B_{ii}$

$$B_{ii} = \lambda_i I + H + \frac{\alpha}{\rho^2} L_{ii} + \frac{\beta}{\rho^3} L_{ii} \quad (35)$$

where  $\alpha$  and  $\beta$  are defined in (14) and (15) and

$$L_{ii} = \begin{pmatrix} l_1^i & l_1^i \\ l_2^i & l_2^i \end{pmatrix} \quad (36)$$

is  $2 \times 2$  matrix (28).

The  $2N \times 2N$  matrix system (9) is equivalent to the set of  $2 \times 2$  matrix systems of the form

$$\frac{1}{\rho} Z' = \mathcal{B} Z \quad (37)$$

where  $\mathcal{B}$  is one of the  $B_{ii}$ .

Now we will construct solutions of the system of ordinary differential equations (37). Let us take

$$Z = \exp\left(\frac{\lambda_i \rho^2}{2}\right) \left( \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} + \left( \frac{\alpha}{\rho^2} + \frac{\beta}{\rho^3} \right) \begin{pmatrix} 0 & 0 \\ -l_1 & -\frac{1}{\rho} l_2 \end{pmatrix} \right) U \quad (38)$$

Then differential equation for  $U$  will be

$$U' = \mathcal{D} U \quad (39)$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & 1 \\ \left(\alpha + \frac{\beta}{\rho^2}\right) l_2 & \left(\alpha + 1 + \frac{\beta}{\rho}\right) (l_1 + l_2) \end{pmatrix} + \dots \quad (40)$$

The last system of equations could be transformed to diagonal form because the determinant of matrix  $\mathcal{D}_0 = \mathcal{D}|_{\rho=\infty} \neq 0$ . The eigen values of matrix

$$\mathcal{D}_0 = \begin{pmatrix} 0 & 1 \\ \alpha l_2 & 0 \end{pmatrix}$$

are  $\nu_{1,2} = \pm \sqrt{\alpha l_2} = \pm \nu$ . Therefore taking the transformation matrix

$$\mathcal{T} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\nu} \\ \frac{1}{2} & -\frac{1}{2\nu} \end{pmatrix}$$

one finds the form of transformed system

$$\begin{aligned} W &= \mathcal{T}U \\ \mathcal{E} &= \mathcal{T}^{-1}\mathcal{D}\mathcal{T} \\ W' &= \mathcal{E}W \end{aligned} \tag{41}$$

with

$$\begin{aligned} \mathcal{E} &= \nu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &+ \frac{1}{\rho} \begin{pmatrix} \frac{\beta}{2\nu}l_2 + \frac{\alpha+1}{2}(l_1 + l_2) & \frac{\beta}{2\nu}l_2 - \frac{\alpha+1}{2}(l_1 + l_2) \\ -\frac{\beta}{2\nu}l_2 - \frac{\alpha+1}{2}(l_1 + l_2) & -\frac{\beta}{2\nu}l_2 + \frac{\alpha+1}{2}(l_1 + l_2) \end{pmatrix} + \\ &\frac{\beta(l_1 + l_2)}{\rho^2} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \dots \end{aligned} \tag{42}$$

Matrices  $\mathcal{E}_{1,2}$  which stand by  $1/\rho$  and  $1/\rho^2$  correspondingly could be transformed to diagonal form as well after one more substitution

$$\begin{aligned} W &= \mathcal{P}Q \\ Q' &= \mathcal{F}Q \end{aligned} \tag{43}$$

Expanding  $\mathcal{P}$  and  $\mathcal{F}$  in powers of  $1/\rho$  one could find the form of matrix  $\mathcal{F}$  solving matrix equation

$$\mathcal{F} = \mathcal{P}^{-1}\mathcal{E}\mathcal{P} - \mathcal{P}^{-1}\mathcal{P}' \tag{44}$$

The result is

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{\rho}\mathcal{F}_1 + \dots \tag{45}$$

where

$$\mathcal{F}_0 = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \tag{46}$$

$$\mathcal{F}_1 = \begin{pmatrix} 0 & \frac{\beta}{2\nu}l_2 + \frac{\alpha+1}{2}(l_1 + l_2) \\ -\frac{\beta}{2\nu}l_2 + \frac{\alpha+1}{2}(l_1 + l_2) & 0 \end{pmatrix} \tag{47}$$

Finally we are able to write down the asymptotic solution of system (43):

$$\begin{aligned} Q_1 &= \exp \left\{ \nu\rho + \left( \frac{\beta}{2\nu}l_2 + \frac{\alpha+1}{2}(l_1 + l_2) \right) \ln \rho \right\} \\ Q_2 &= \exp \left\{ -\nu\rho + \left( -\frac{\beta}{2\nu}l_2 + \frac{\alpha+1}{2}(l_1 + l_2) \right) \ln \rho \right\} \end{aligned} \tag{48}$$

where

$$\begin{aligned}\nu &= \sqrt{\alpha}\sqrt{l_2} \\ \alpha &= \pm \frac{(2N)!}{4(i\zeta)^{2N}} \left( (1-\mu)^2 - \frac{M^2}{m^2} \right) \\ \beta &= \pm \frac{(2N)!}{8(i\zeta)^{2N}} (1+\mu) \left( (2(1-\mu)^2 - \frac{M^2}{m^2}) \right)\end{aligned}\quad (49)$$

Using this expression we could restore the asymptotic solution of initial matrix system of differential equations (9) and therefore the asymptotic solution of equation (1) could be written in the form

$$\Psi(\rho) = \rho^a \Phi_1(\rho) + \rho^b \Phi_2(\rho) \quad (50)$$

where  $\Phi_{1,2}(\rho)$  are some one-valued functions and  $a, b$  are given by the expressions

$$\begin{aligned}a &= \left( -\frac{\beta}{2\nu} l_2 + \frac{\alpha+1}{2} (l_1 + l_2) \right) \\ b &= \left( \frac{\beta}{2\nu} l_2 + \frac{\alpha+1}{2} (l_1 + l_2) \right)\end{aligned}\quad (51)$$

The coefficients  $l_1$  and  $l_2$  (see (28) and (36)) are expressed through the elements  $\tilde{t}_{ij}$  of the matrix inverse to  $T$  ((24) and (27)). Therefore in order to determine the asymptotic expression for solution of (1) we just need to calculate the inverse matrix  $T^{-1}$ .

Matrix  $T$  has the form

$$T = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_1 + \frac{1}{2} & \lambda_2 & \dots & \lambda_N + \frac{1}{2} \\ \lambda_1^2 & \lambda_1^2 + \lambda_1 & \lambda_2^2 & \dots & \lambda_N^2 + \lambda_N \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda_1^{2N-1} & \lambda_1^{2N-1} + \frac{2N-1}{2} \lambda_1^{2N-2} & \lambda_2^{2N-1} & \dots & \lambda_N^{2N-1} + \frac{2N-1}{2} \lambda_N^{2N-2} \end{pmatrix} \quad (52)$$

Its determinant will not change if we subtract odd columns from even ones:

$$\det T = \det T_1 = \begin{vmatrix} 1 & 0 & 1 & \dots & 0 \\ \lambda_1 & \frac{1}{2} & \lambda_2 & \dots & \frac{1}{2} \\ \lambda_1^2 & \lambda_1 & \lambda_2^2 & \dots & \lambda_N \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda_1^{2N-1} & \frac{2N-1}{2} \lambda_1^{2N-2} & \lambda_2^{2N-1} & \dots & \frac{2N-1}{2} \lambda_N^{2N-2} \end{vmatrix} \quad (53)$$

The last expression could be written in the following form

$$\det T = \frac{1}{2^N} \partial_{\mu_1} \dots \partial_{\mu_N} W|_{\mu_i = \lambda_i} \quad (54)$$

where  $W$  is Wandermund determinant

$$W = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \mu_1 & \lambda_2 & \dots & \mu_N \\ \lambda_1^{2N-1} & \mu_1^{2N-1} & \lambda_2^{2N-1} & \dots & \mu_N^{2N-1} \end{vmatrix} \quad (55)$$

Using the well known rule of calculation of Wandermund determinant we obtain for  $\det T$

$$\det T = \frac{1}{2^N} \prod_{i>j} (\lambda_i - \lambda_j)^4 \quad (56)$$

Implementing the same trick for the calculation of corresponding minors  $M_{ij}$  we obtain the expression for the elements of inverse matrix  $T^{-1}$ :

$$\begin{aligned} \tilde{t}_{1N} &= 2 \left( \frac{1}{\prod_{k=2}^N (\lambda_k - \lambda_1)^2} - \sum_j \frac{1}{\lambda_j - \lambda_1} \prod_{k=2}^N \frac{1}{(\lambda_k - \lambda_1)^2} \right) \\ \tilde{t}_{2N} &= -2 \prod_{k=2}^N \frac{1}{(\lambda_k - \lambda_1)^2} \end{aligned} \quad (57)$$

Then

$$\begin{aligned} l_2 &= -2 \prod_{k=2}^N \frac{1}{(\lambda_k - \lambda_1)^2} \\ l_1 + l_2 &= -2 \sum_{j=2}^N \frac{1}{\lambda_j - \lambda_1} \prod_{k=2}^N \frac{1}{(\lambda_k - \lambda_1)^2} \end{aligned} \quad (58)$$

The product and the sum entering last equation could be evaluated using the characteristic equation (20) for matrix  $A_0$  (11). Indeed, if one write the characteristic polynomial  $P_{2N}$  in the form

$$P_{2N} = (\lambda - \lambda_1) \dots (\lambda - \lambda_{2N}) \quad (59)$$

which for  $N$  large enough become approximately

$$P_{2N} \approx (\lambda - \lambda_1)^2 \dots (\lambda - \lambda_N)^2 \quad (60)$$

because all the eigen values of matrix  $A_0$  become twice degenerate, as it is explained earlier, then

$$P''_{2N}|_{\lambda=\lambda_1} \approx (\lambda_1 - \lambda_2)^2 \dots (\lambda_1 - \lambda_N)^2 \quad (61)$$

exactly the required product. Taking into account expression (22) for  $P_{2N}$  in the limit  $N \rightarrow \infty$  one easily finds

$$\prod_{k=2}^N (\lambda_k - \lambda_1)^2 = \frac{(2N)!}{(i\zeta)^{2N}} \left( \frac{\zeta^2}{4} \right) \cos \left\{ \frac{\zeta \lambda_1}{2} \right\} \quad (62)$$

In a similar way

$$\sum_{k=2}^N \frac{1}{\lambda_k - \lambda_1} = - \frac{\sum_{k=2}^N \prod_{m=2, m \neq k}^N (\lambda_1 - \lambda_m)}{\prod_{k=2}^N (\lambda_1 - \lambda_k)} = - \sqrt{\left. \frac{P_{2N}^{(4)}}{P_{2N}^{(2)}} \right|_{\lambda=\lambda_1}} \quad (63)$$

and then

$$\sum_{k=2}^N \frac{1}{\lambda_k - \lambda_1} = \pm \frac{\zeta}{2\sqrt{2}} \quad (64)$$

If one substitute these expressions into (51) one finds that the powers of  $\rho$  which describe the branching of solution at infinity become

$$\begin{aligned} a_{\pm} &= - \frac{2(1-\mu)^2 - M^2/m^2}{\zeta\sqrt{2}\sqrt{(1-\mu)^2 - M^2/m^2}} \pm \frac{1}{\zeta 2\sqrt{2}} \left( (1-\mu^2)^2 - M^2/m^2 \right) \\ b_{\pm} &= \frac{2(1-\mu)^2 - M^2/m^2}{\zeta\sqrt{2}\sqrt{(1-\mu)^2 - M^2/m^2}} \pm \frac{1}{\zeta 2\sqrt{2}} \left( (1-\mu^2)^2 - M^2/m^2 \right) \end{aligned} \quad (65)$$

The plus or minus signs are taken at infinities in  $S_+$  and  $S_-$  components of  $S_F$  correspondingly.

### 3 Mass spectrum

Having determined the form (50) of asymptotic solutions of equation (1) at infinities in  $S_+$  and  $S_-$  components of the Riemannian surface  $S_F$  we can try to impose the condition of decreasing of the wave function at both infinities. But the asymptotic expression for the solution (50) contains branching functions

if powers  $a$  and  $b$  are not integer. This means that the asymptotic expression (50) could not be a good approximation for the one valued wave function  $\Psi$ . It is known from complex analysis, that if the powers  $a$  and  $b$  of  $\rho$  in asymptotic expression (50) for the wave function  $\Psi$  are not integer numbers, the approximation (50) is valid only in some sector  $\phi_1 < \text{Arg}(1/\rho) < \phi_2$  in the neighbourhood of infinity point. Different approximations are valid in different overlapping sectors. This behaviour of the asymptotic approximation is called Stokes phenomenon [5]. But if both  $a$  and  $b$  are integers, the expression (50) is a good approximation for non-branching wave function everywhere in the punctured neighbourhoods of infinities.

Due to the double degeneracy of eigen values of matrix  $A_0$  the infinite matrix equation is reduced to the infinite set of second order matrix equations. This second order equations can be analysed in the same way as we do this in nonrelativistic quantum mechanics. And we conclude therefore that the wave function with needed boundary conditions on the both sides of the Riemannian surface exists only in the situation described above - when  $a_{\pm}$  and  $b_{\pm}$  are integers.

The conditions for  $a, b$  to be integers are equivalent to

$$\begin{aligned} \frac{2(1-\mu)^2 - M^2/m^2}{\zeta\sqrt{2}\sqrt{(1-\mu)^2 - M^2/m^2}} &= \mathbf{n} \\ \frac{1}{\zeta 2\sqrt{2}} \left( (1-\mu)^2 - M^2/m^2 \right) &= \mathbf{k} \end{aligned} \quad (66)$$

where  $\mathbf{n}$  and  $\mathbf{k}$  are integers. These conditions define the mass spectrum for the equation (1).

The motion of dust shell in its own gravitational field is parameterised by three parameters – the bare mass of the shell  $M$ , the Schwarzschild mass which is measured at “right” infinity  $m = m_{out}$  and the Schwarzschild mass inside the shell (or the mass at “left” infinity)  $m_{in} = \mu m$ . There are two equations (66) on these three parameters, which we could solve for two of them. Therefore, the spectrum of Schwarzschild mass  $m$  is parameterised by two discrete parameters  $\mathbf{n}$  and  $\mathbf{k}$  and one continuous (for example  $\mu$ ). But if we restrict ourselves with situations when the gravitational field in which the shell moves is produced only by the shell itself, that is there is no space-time singularity inside the shell, then  $\mu = 0$  and the mass spectrum of such states is discrete.

The expression (66) for the mass spectrum is valid only when  $(1 - \mu)^2 - M^2/m^2 > 0$  when the square root expression entering (66) is well defined. This restriction corresponds to classical restriction on the trajectories of bound motion of the shell. If it is satisfied, then there exist some maximal radius of expansion of the shell  $\rho(t) < \rho_{max}$ .

Otherwise the shell could escape to infinity in its classical motion. For such shells first of conditions (66) disappears and the only left is

$$\frac{1}{\zeta 2\sqrt{2}} \left( (1 - \mu)^2 - M^2/m^2 \right) = \mathbf{k} \quad (67)$$

It is remarkable that for light-like shells ( $M = 0$ ) in their own gravitational field ( $\mu = 0$ ) this quantization condition become (we recall that  $\zeta = m_{pl}^2/2m^2$ )

$$m = m_{pl}\sqrt{2\mathbf{k}} \quad (68)$$

## 4 Conclusion

In this technical note we developed a method of constructing asymptotical solutions (50) of relativistic Schroedinger equation in finite differences (1), which describe a self-gravitating spherically symmetric dust shell of matter [1]. This finite difference equation is defined over Riemannian surface  $S_F$  (4). The configuration space of the system is the real section of this surface and has nontrivial geometry. This complicated geometry of configuration space of quantum system is due to complicated geometric structure of classical space-time manifold of self-gravitating shell. Analysis of the behaviour of asymptotic solution on Riemannian surface  $S_F$  enabled us to write down quantization conditions (66) under which the one valued solution of (1) with physical boundary conditions exists. These quantization conditions define discrete spectrum for Schwarzschild mass  $m$  if the mass inside the shell is zero. The main result of this paper is that this spectrum up to the numerical factors coincides with the spectrum, found in [1] in large black hole approximation and in [2] in quasiclassical approximation. This discrete mass spectrum is of the form

$$m \sim m_{pl}\sqrt{\mathbf{k}} \quad (69)$$

for the light-like shells (shells with bare mass  $M = 0$ ). This form of quantum black hole spectrum was extensively discussed by many authors [6, 7].

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